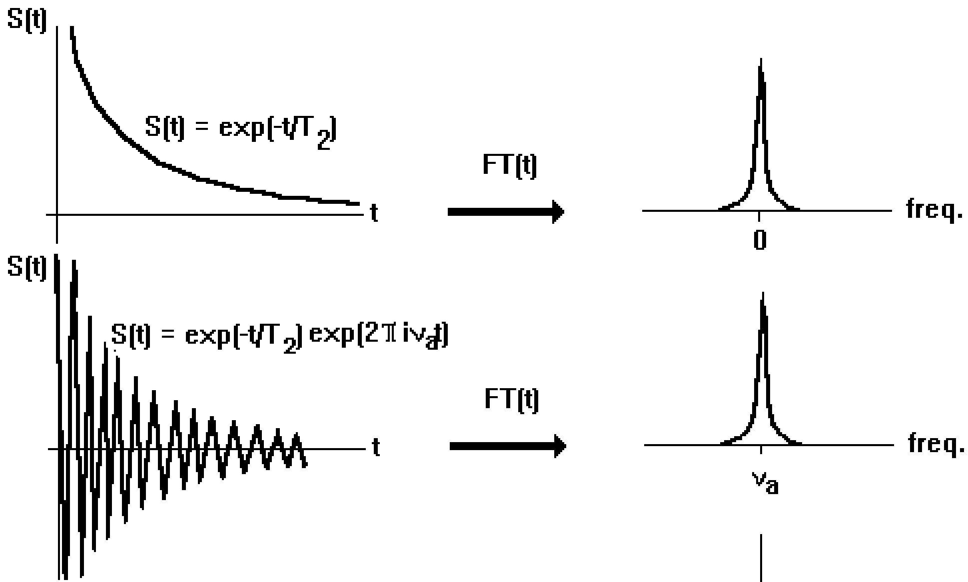
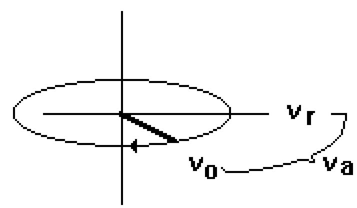
FOURIER TRANSFORMATION and PHASE CORRECTION

time domain signal S =S(t) "on resonance" and "off resonance"





ν_r = rotating frame frequency

ν_o = Larmor frequency

ν_a = audio frequency

memo 1 : FT ---> $S(\upsilon) = \int S(t) \; \exp(-i2\pi\upsilon t)\, dt$ $2\pi\upsilon = \omega$

memo 2 : IFT ---> $S(t) = \int S(\upsilon) \exp(+i2\pi\upsilon t)\, d\upsilon$

memo 3 : S(υ) and S(t) are Fourier pairs

$$S(t) = \exp(-t/T_2) \exp(i2\pi\upsilon_{\mathbf{a}} t)$$

$$S(\upsilon) = \int S(t) \exp(-i2\pi\upsilon t)\, dt$$

$$\Downarrow$$

$$S(\upsilon) = \int \exp(-t/T_2) \exp(i2\pi\upsilon_{\mathbf{a}} t) \exp(-i2\pi\upsilon t)\, dt$$

$$= \int \exp\{(-1/T_2 + i2\pi[\upsilon_{\mathbf{a}} - \upsilon])t\}\, dt$$

$$= \int \exp\{-t\, a\}\, dt \qquad \text{if } a = 1/T_2 + i2\pi[\upsilon - \upsilon_{\mathbf{a}}]$$

memo 4 : $S(\upsilon) = \int_0^\infty \exp\{-a\, t\}\, dt = \left[1/a \exp(-a\, t)\right]_\infty^0$

if $t = 0$ then $S(\upsilon)=1/a$

if $t = \infty$ then $S(\upsilon) = 0$

therefore : $S(\upsilon)=1/a - 0 =$ **$1/a$**

the solution is : $S(\upsilon) = 1/(1/T_2 + i2\pi[\upsilon - \upsilon_{\mathbf{a}}])$

to separate real from imaginary part :

$$S(\upsilon) \quad = \; 1/(1/T_2 + i2\pi[\upsilon - \upsilon_{\mathbf{a}}]) \qquad\qquad * \text{ and / by } \{1/T_2 - i2\pi[\upsilon - \upsilon_{\mathbf{a}}]\}$$

$$S(\upsilon) \quad = \; (1/T_2 - i2\pi[\upsilon - \upsilon_{\mathbf{a}}])/\{(1/T_2 + i2\pi[\upsilon - \upsilon_{\mathbf{a}}])(1/T_2 - i2\pi[\upsilon - \upsilon_{\mathbf{a}}])\}$$

memo 5 : $1/(a+ib) = (a-ib)/\{(a+ib)(a-ib)\}$

$= (a-ib)/(a^2+b^2)\}$

$$S(\upsilon) \quad = \; (1/T_2 - i2\pi[\upsilon - \upsilon_{\mathbf{a}}])/\{1/T_2{}^2 - 4\pi^2[\upsilon - \upsilon_{\mathbf{a}}]^2\}$$

$$= \; 1/T_2\,/\{1/T_2{}^2 - 4\pi^2[\upsilon - \upsilon_{\mathbf{a}}]^2\} + \; i\,\{-2\pi[\upsilon - \upsilon_{\mathbf{a}}])\}/\{1/T_2{}^2 - 4\pi^2[\upsilon - \upsilon_{\mathbf{a}}]^2\}$$

---------------A(υ)---------- + i --------------D (υ)---------------------

absorptive signal *dispersive signal*

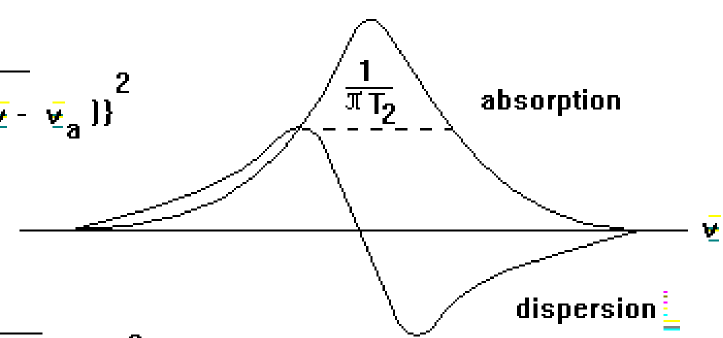
The signal has the following form:

S(υ) = A(υ) + i D (υ)

where the real and the imaginary parts are separated.

$$A(\nu) = \frac{\frac{1}{T_2}}{\left(\frac{1}{T_2}\right)^2 + \{2\pi(\nu - \nu_a)\}^2}$$

$$D(\nu) = \frac{-2\pi(\nu - \nu_a)}{\left(\frac{1}{T_2}\right)^2 + \{2\pi(\nu - \nu_a)\}^2}$$



PHASE CORRECTION

A: *the phase correction of the* {$\exp(-t/T_2)\ \exp(i2\pi\upsilon_{\mathbf{a}}t)$} *complex function*

$S(t)' \quad = \exp(-t/T_2)\ \exp(i2\pi\upsilon_{\mathbf{a}}t)$

$S(t) \quad = \exp(+i\phi)\ S(t)'$ $\qquad$ $\exp(+i\phi)$ is a constant $\quad$ (ϕ = the phase correction factor)

$S(t) \quad = \exp(-t/T_2)\ \exp(i2\pi\upsilon_{\mathbf{a}}t)\ \exp(+i\phi)$

after FT:

$$S(\upsilon) = \exp(+i\phi) \quad * \quad \{A(\upsilon) + i\,D(\upsilon)\}$$

$$= \{\cos(\phi) + i\sin(\phi)\} * \{A(\upsilon) + i\,D(\upsilon)\}$$

$$= \{\cos(\phi)\,A(\upsilon) - \sin(\phi)\,D(\upsilon)\} + i\,\{\sin(\phi)\,A(\upsilon) + \cos(\phi)\,D(\upsilon)\}$$

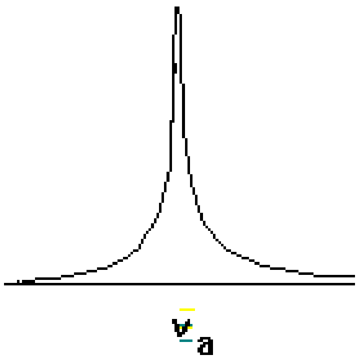
--------------------R(υ)------------ + i -------------I(υ)-----------------

comment : both the real (R(υ)) and the imaginary (I(υ)) parts have absorptive and dispersive character

The phase correction of $S(\upsilon)$ is the lin. comb. of the real and of the imaginary part.

$$R(\upsilon) \cos(\phi) + I(\upsilon) \sin(\phi) =$$
$$= \{\cos(\phi) A(\upsilon) - \sin(\phi) D(\upsilon)\} \cos(\phi) + \{\sin(\phi) A(\upsilon) + \cos(\phi) D(\upsilon)\} \sin(\phi)$$
$$= \cos^2(\phi) A(\upsilon) - \sin(\phi)\cos(\phi) D(\upsilon) + \sin^2(\phi) A(\upsilon) + \sin(\phi)\cos(\phi) D(\upsilon)$$
$$= A(\upsilon) \{ \cos^2(\phi) + \sin^2(\phi)\}$$
$$= A(\upsilon)$$

After phase correcting a complex signal
a pure absorptive curve is obtained: **$A(\upsilon)$**



comment : Phase correction is a multiplication by $\exp(-i\phi)$

The aim is to correct the instrumental delay that resulted in the $\exp(+i\phi)$ term.

$$S(t) = \exp(+i\phi)\, S(t)'\, \mathbf{\exp(-i\phi)}$$

comment : the phase correction may not be constant for all peaks of the spectrum

$\phi = \phi_o + (n/N)\, \phi_1$

ϕ_o = zero order phase correction $0^o \leq ... \leq 360^o$
originates from the phase diff. of the transmitter and receiver

ϕ_1 = first order phase correction $0^o \leq ... \leq \infty^o$
originates form electronic delay and from the pulsing time

and

$0 \leq n \leq N$ n = spectrum data point

N = max. value of the SW

B: *the phase correction of a real signal* $\exp(-t/T_2) \cos(2\pi\upsilon_{\mathbf{a}}t + \phi)$

$$S(t) = \exp(-t/T_2) \cos(2\pi\upsilon_{\mathbf{a}}t + \phi)$$

memo 7 : $\exp(ix) = \cos(x) + i \sin(x)$

$+ \quad \exp(-ix) = \cos(x) - i \sin(x)$

$\exp(ix) + \exp(-ix) = 2\cos(x)$

$1/2\{\exp(ix) + \exp(-ix)\} = \cos(x)$

$$\begin{aligned} S(t) &= \exp(-t/T_2) \cos(2\pi\upsilon_{\mathbf{a}}t + \phi) \\ &= \exp(-t/T_2)\, 1/2\{ \exp(i (2\pi\upsilon_{\mathbf{a}}t + \phi)) + \exp(-i (2\pi\upsilon_{\mathbf{a}}t + \phi)) \} \\ &= 1/2 \exp(-t/T_2)\{ \exp(i\, 2\pi\upsilon_{\mathbf{a}}t) \exp(i\phi) + \exp(-i\, 2\pi\upsilon_{\mathbf{a}}t) \exp(-i\phi) \} \\ &= \underbrace{1/2 \exp(-t/T_2) \exp(i\, 2\pi\upsilon_{\mathbf{a}}t) \exp(i\phi)}_{S_1(t)} + \underbrace{1/2 \exp(-t/T_2) \exp(-i\, 2\pi\upsilon_{\mathbf{a}}t) \exp(-i\phi)}_{S_2(t)} \end{aligned}$$

------------------$S_1(t)$------------------ + --------------$S_2(t)$------------------------------

memo 8 : $FT(f + g) = FT(g) + FT(f)$

$S_1(t) \quad = 1/2 \exp(-t/T_2) \exp(i\, 2\pi\upsilon_{\mathbf{a}} t) \exp(i\phi)$

$S_2(t) \quad = 1/2 \exp(-t/T_2) \exp(-i\, 2\pi\upsilon_{\mathbf{a}} t) \exp(-i\phi)$

$\Downarrow$ FT (as above for the complex function)

$S_1(\upsilon) \quad = 1/2 \{\cos(\phi) A(\upsilon) - \sin(\phi) D (\upsilon)\} + i \{\sin(\phi) A(\upsilon) + \cos(\phi) D (\upsilon)\}$

$S_2(\upsilon) \quad = 1/2 \{\cos(\phi) A(-\upsilon) +\sin(\phi) D (-\upsilon)\} + i \{-\sin(\phi) A(-\upsilon) + \cos(\phi) D (-\upsilon)\}$

--------------------R(υ)------------ + i -------------I(υ)-----------------

$S(\upsilon) \quad = S_1(\upsilon) + S_2(\upsilon)$

$S(\upsilon) \quad = 1/2 \{\cos(\phi) [A(\upsilon) + A(-\upsilon)] - \sin(\phi) [D(\upsilon) - D(-\upsilon)]\} +$ (R(υ) real part)

$+ 1/2\, i\{\cos(\phi) [D(\upsilon) + D(-\upsilon)] + \sin(\phi) [A(\upsilon) - A(-\upsilon)]\}$ (I(υ) imaginary part)

The phase correction of S(υ) is the lin. comb. of the real and of the imaginary part.

$R(\upsilon) \cos (\phi) + I(\upsilon) \sin (\phi) =$

$= 1/2 \{\cos(\phi) [A(\upsilon) + A(-\upsilon)] - \sin(\phi) [D(\upsilon) - D(-\upsilon)]\}\cos (\phi) +$

$1/2 \{\cos(\phi) [D(\upsilon) + D(-\upsilon)] + \sin(\phi) [A(\upsilon) - A(-\upsilon)]\}\sin (\phi) =$

$= 1/2 \{\cos^2(\phi) A(\upsilon) + \cos^2(\phi) A(-\upsilon) - \sin(\phi)\cos (\phi) D(\upsilon) + \sin(\phi)\cos (\phi) D(-\upsilon)+$

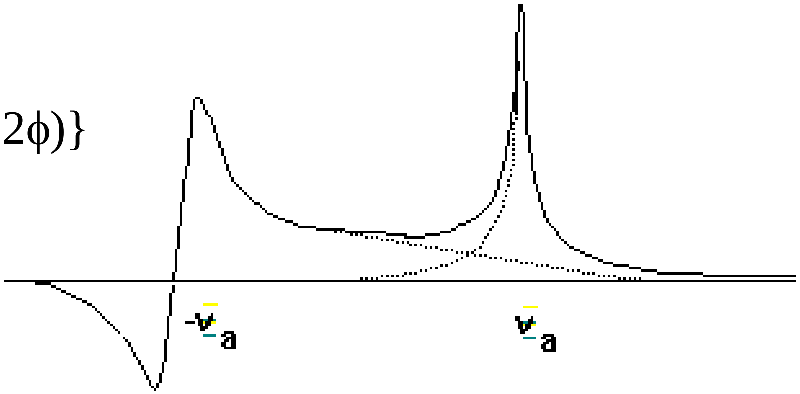
$\sin(\phi)\cos(\phi) D(\upsilon) + \sin(\phi)\cos(\phi) D(-\upsilon) + \sin^2(\phi) A(\upsilon) - \sin^2(\phi) A(-\upsilon)\} =$

$= 1/2 \{A(\upsilon) \{\cos^2(\phi) + \sin^2(\phi)\} + A(-\upsilon) \{\cos^2(\phi) - \sin^2(\phi)\} + D(\upsilon)\, 2 \sin(\phi)\cos (\phi)\}$

$=1/2 \{A(\upsilon) \quad + A(-\upsilon) \cos(2\phi) \quad + D(\upsilon) \sin(2\phi)\}$

So the phase corrected real signal is :

$$1/2 \ \{A(\upsilon_a) + A(-\upsilon_a)\cos(2\phi) + D(\upsilon_a)\sin(2\phi)\}$$



- a pure absorptive at υ_a

- a "mixed mode" at $-\upsilon_a$

(the sum of a cos modulated absorptive and a sin modulated dispersive)

problems:

1. in case of a real signal we obtain two peaks

2. signal at υ_a is absorptive (o.k.) but the signal at $-\upsilon_a$ has a mixed phase.

3. the "tailing" of the mixed mode signal at $-\upsilon_a$ can initiate base line distortions even at $+\upsilon_a$.

conclusion : obtaining "only" a real signal ---> no possibility to discriminate between positive and negative signals.

aim : only if one could record a complex signal?

solution : quadrature detection.

THE QUADRATURE DETECTION

initial technical comment:

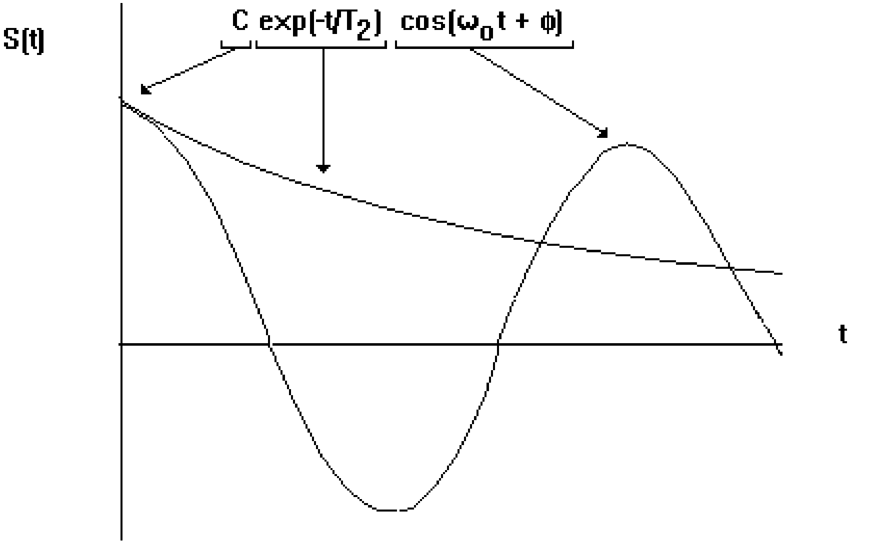
- with one carrier reference only real signals can be recorded.

- with two carrier references (90^o out of phase) complex signals can be recorded.

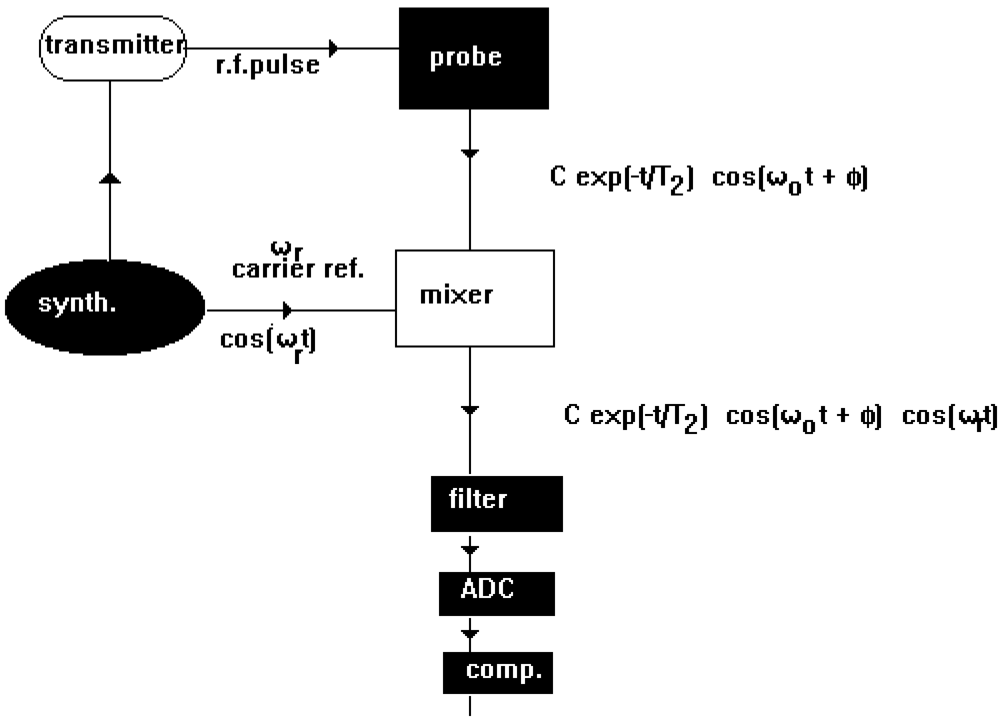
ω_r = rotating frame frequency
ω_o = Larmor frequency
ω_a = audio frequency

A : Normal detection scheme (one carrier reference)

the signal is: $S(t) = C^*\exp(-t/T_2) \cos(\omega_o t + \phi)$



the block diagram of a "single carrier reference" detection system:



the modulated "real" signal is : $S'(t) = C*\exp(-t/T_2)\cos(\omega_o t + \phi)\cos(\omega_r t)$

memo 9 : $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$+ \quad \cos(a-b) = \cos(a)\cos(-b) - \sin(a)\sin(-b)$

$\cos(a+b) + \cos(a-b) = 2\cos(a)\cos(b)$ / $\cos(-b) = \cos(+b)$ and

$\sin(-b) = -\sin(+b)$

if $\omega_o t + \phi = a$ and $\omega_r t = b$

then $S'(t) = 1/2\, C \exp(-t/T_2) \{2 \cos(\omega_o t + \phi) \cos(\omega_r t)\}$

$= 1/2\, C \exp(-t/T_2) \{2 \cos([\omega_o + \omega_r]t + \phi) \cos([\omega_o - \omega_r]t + \phi)\}$

$[\omega_o + \omega_r]$ ---> high frequency ---> rejected

$[\omega_o - \omega_r]$ ---> low frequency (audio frequency) ---> stored

the signal : $= 1/2\, C \exp(-t/T_2) \{2 \cos([\omega_o - \omega_r]t + \phi)$ $/\omega_o - \omega_r = \omega_a$

$= 1/2\, C \exp(-t/T_2) \{2 \cos([\omega_a]t + \phi)$ $/2\pi\upsilon_a = \omega_a$

$= C \exp(-t/T_2) \cos(2\pi\upsilon_a t + \phi)$

this signal is phased as mentioned previously (c.f. the phase correction of a real signal)

B : Quadrature detection scheme (two carrier reference)

the cos(ωt) modulated "real" signal is :

$S'(t) = C^*\exp(-t/T_2) \cos(\omega_o t + \phi) \cos(\omega_r t)$

as seen above the "real" signal before phasing:

$S'(t) = C \exp(-t/T_2) \cos([\omega_o - \omega_r]t + \phi)$

the sin(ωt) modulated "real" signal is :

$S'(t) = C^*\exp(-t/T_2) \cos(\omega_o t + \phi) \sin(\omega_r t)$

memo 10 :

$\cos(a) \sin(b) = 1/2[\sin(a + b) - \sin(a - b)]$

if $\omega_o t + \phi = a$ and $\omega t = b$

then S '(t) $= 1/2\ C \exp(-t/T_2)\ \{2 \cos(\omega_o t + \phi) \sin(\omega_r t)\}$

$= 1/2\ C \exp(-t/T_2)\ \{2 \sin([\omega_o + \omega_r]t + \phi) \sin([\omega_o - \omega_r]t + \phi)\}$

rejecting the $[\omega_o + \omega]$ the signal is:

$$S''(t) = C \exp(-t/T_2) \sin([\omega_o - \omega_r]t + \phi)$$

If we consider S '(t) as the **real** and S "(t) as the **imaginary** part of the signal, then the complex signal is :

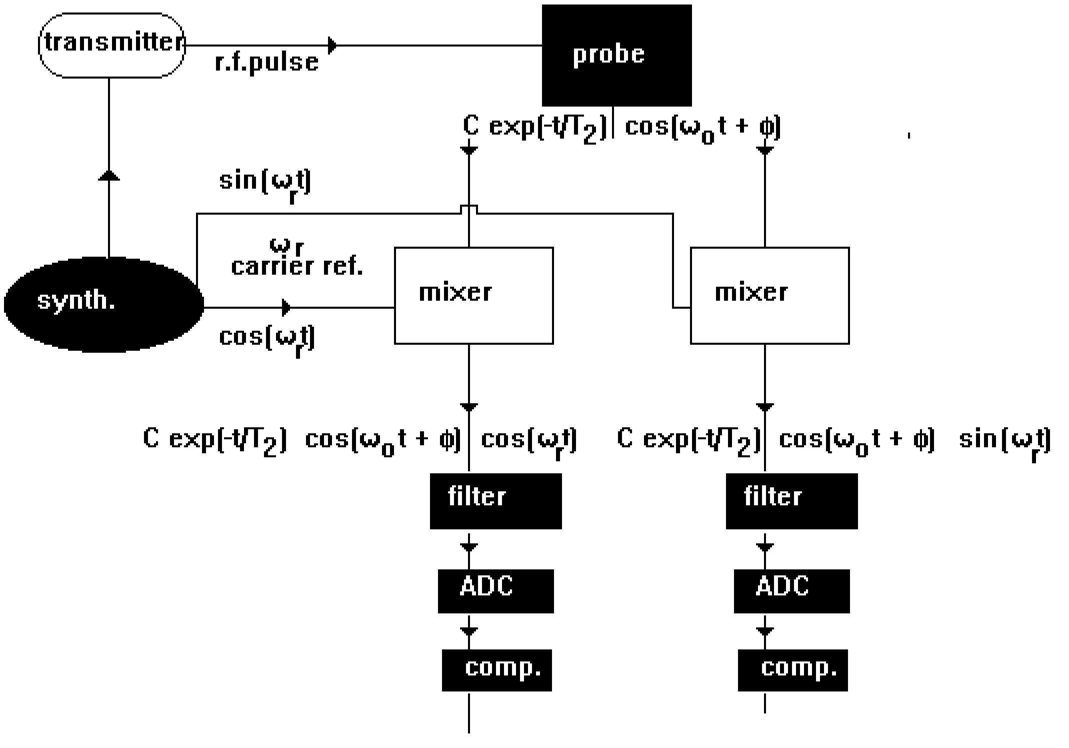
$$C \exp(-t/T_2)\ \{ \cos([\omega_a]t + \phi) + i\ \sin([\omega_a]t + \phi) \}$$

$$C \exp(-t/T_2)\ \exp(\omega_a t + \phi) \qquad /2\pi\upsilon_a = \omega_a$$

$$C \exp(-t/T_2)\ \exp(2\pi i \upsilon_a t + \phi)$$

This "complex" signal can be phased as mentioned above.

the block diagram of a quadrature detection system:



A numerical example

ω_r = rotating frame frequency = 500 000 000 Hz (500 MHz)
ω_o = Larmor frequency = 500 000 100 Hz
ω_a = audio frequency = 100 Hz (the off set)